Higher-Order Random Walkers: Blob Diffusion on Complex Networks

Moritz Laber[1,2,★] Harrison Hartle[1], Guillaume St-Onge[1], and Brennan Klein[1]
1 Network Science Institute, Northeastern University, Boston, MA, United States
2 Complexity Science Hub, Vienna, A-1080, Austria
★ laber.m@northeastern.edu

What is a blob?

Random walks [1] in which a single random walker occupies one node at a time are one of the most fundamental models of dynamics on networks. A **blob walker** B_t of size b occupies not only a single node but a connected subgraph of size b at time t.

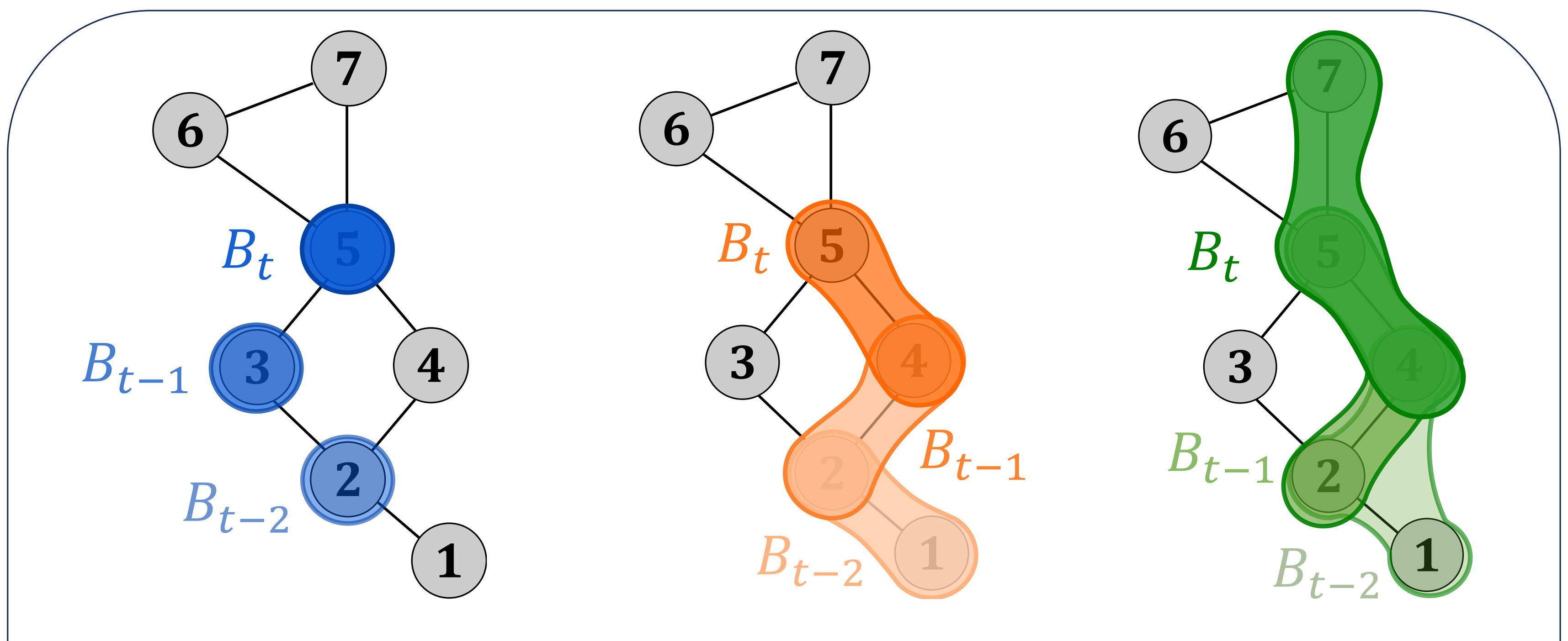

Figure 1: The blobs of size $b = 1$ (left, blue) $b = 2$ (center, orange) and $b = 3$ (right, green) are displayed on the same graph G at three different times $t-1, t-2, t$ (indicated by fading).

How does a blob move?

Each time step t the blob walker $B_t = \{v_1, \ldots, v_b\}$ consisting of vertices v_1 to v_b transitions to a new configuration B_{t+1} such that

- B_{t+1} is again a connected subgraph.
- B_{t+1} overlaps in $b-1$nodes with B_t i.e., $|B_t \cap B_{t+1}| = b-1$.

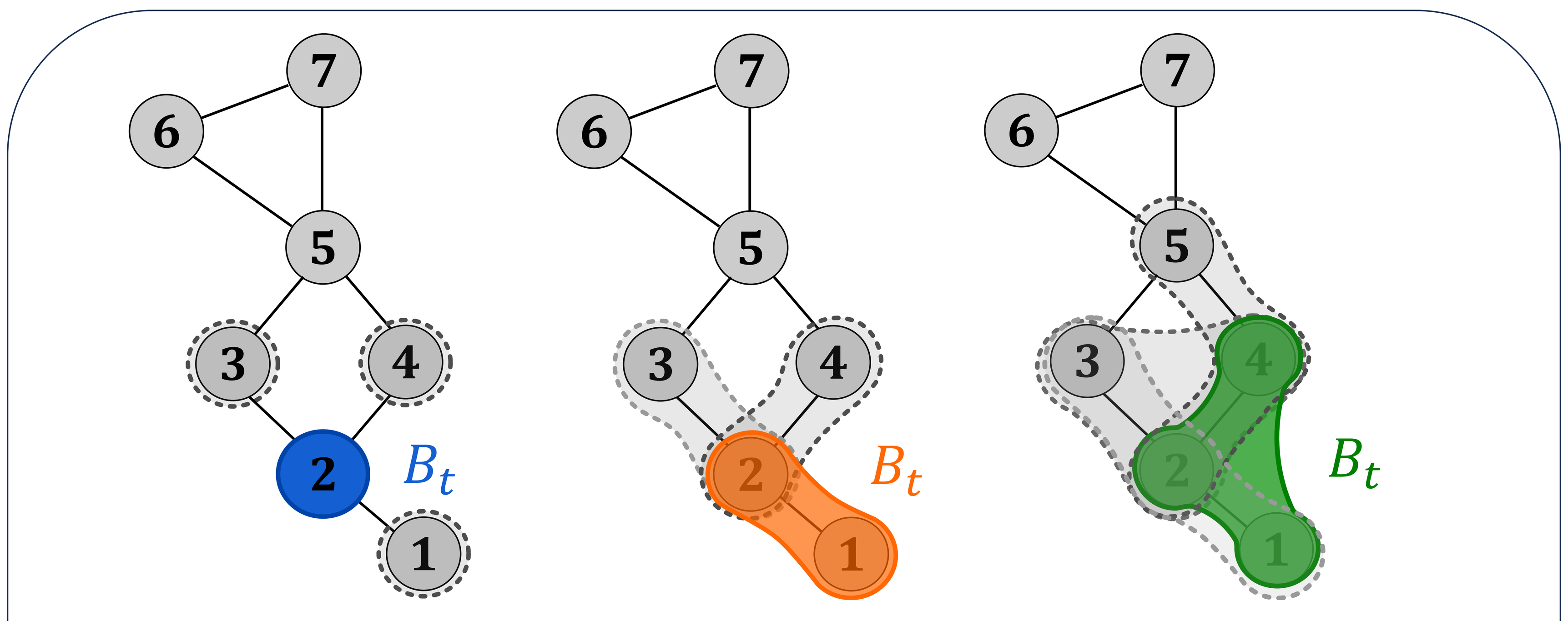

Figure 2: Possible next configurations (dashed) for a blob in configuration B_t *Left*: $b = 1$: $B_t = \{2\}$ could transition into configurations $\{1\}, \{3\}$ or $\{4\}$ at $t+1$. *Center*: $b = 2$: $B_t = \{1,2\}$ can move to $\{2,3\}$ or $\{2,4\}$. *Right*: $B_t = \{1,2,4\}$ could move to $\{1,2,3\}$, $\{2,3,4\}$, or $\{2,4,5\}$.

b- Graphs: Generalized Line Graphs

Given a graph G, we define the ***b*-graph** G_b as the graph consisting of connected subgraphs of G as nodes, that are linked if they overlap in $b-1$ nodes. The b-graph is the state space of the Markov chain describing the movement of a blob of size b. G_1 is isomorphic to G and G_2 is the linegraph of G. For $b \geq 3$, G_b is a generalization of the linegraph.

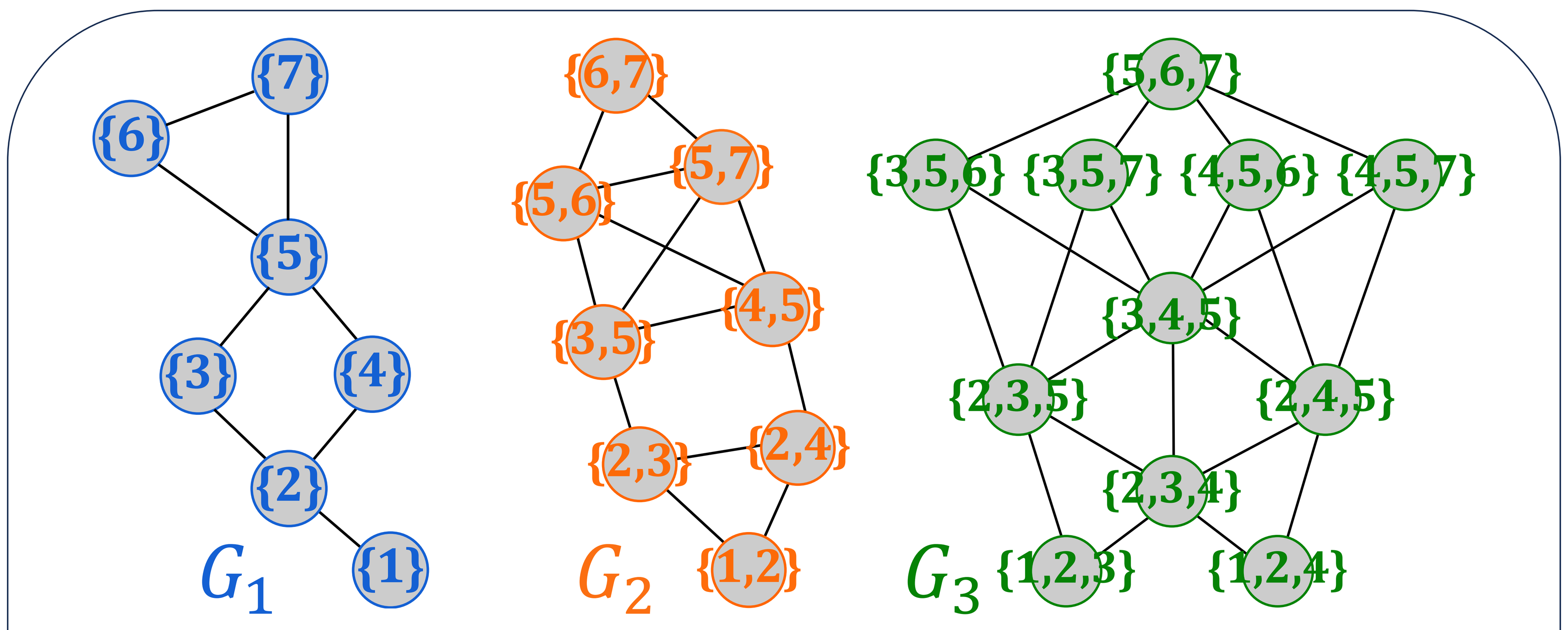

Figure 3: The b-graph of the graph G in Fig. 1 and Fig. 2. *Left:* The 1-graph G_1 is isomorphic to the graph G. *Center:* The 2-graph is the linegraph of G. *Right:* The 3-graph is a generalization of the linegraph , consisting of connected subgraphs of size 3 as nodes that are linked if they share 2 of 3 nodes.

Stationary Distribution and b-Centrality

We can define a family of centrality measure $c_i^{(b)}$ for a node i of G by summing the degree of all nodes in the b-graph G_b that represent subgraphs that contain node i. We call this centrality ***b*-centrality** and derive an analytical expression for $b \leq 3$. The b-centrality is proportional to the stationary distribution of a blob walk of size b (Fig 4) . It scales as the bth power of the degree k for large $k \gg 1$ but can lead to very different node rankings for low k (Fig. 5).

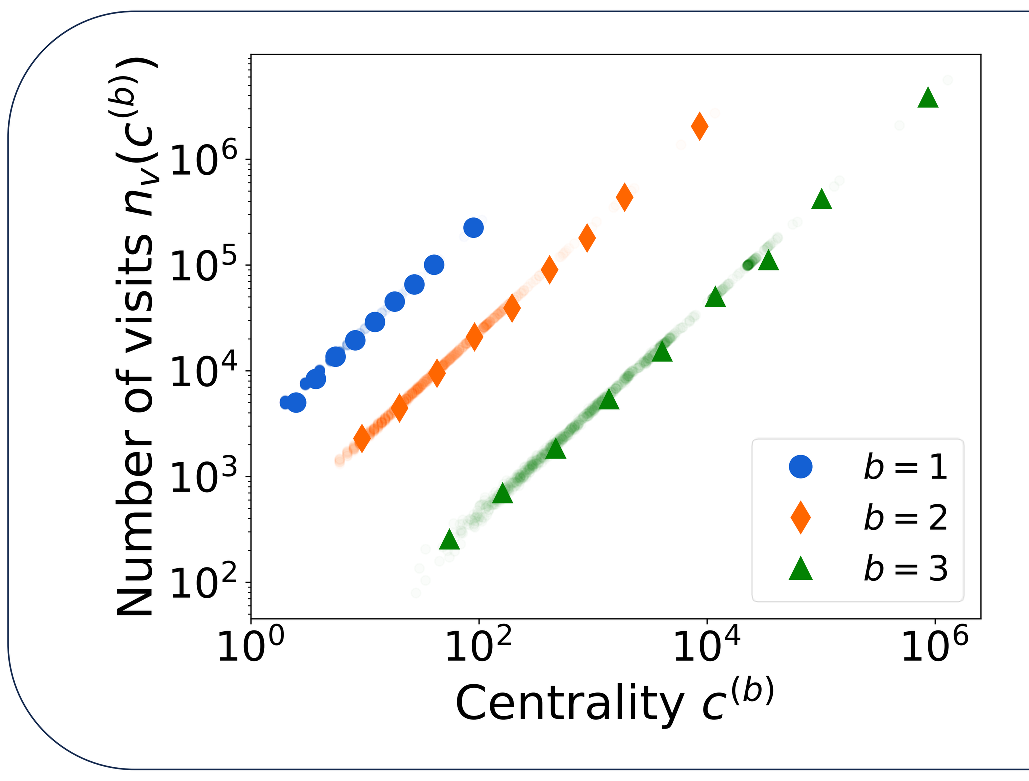

Figure 4: The b-centrality $c_i^{(b)}$ of node i determines the number of times n_v a blob of size b visits node i. The centrality $c_i^{(b)}$ is calculated analytically. Visits are obtained by numerical simulation of $T = 10^6$ steps on a Barabási-Albert network with $n = 10^3$ nodes and $m = 2 \cdot 10^3$ edges. Filled symbols indicate binning, transparent ones raw data.

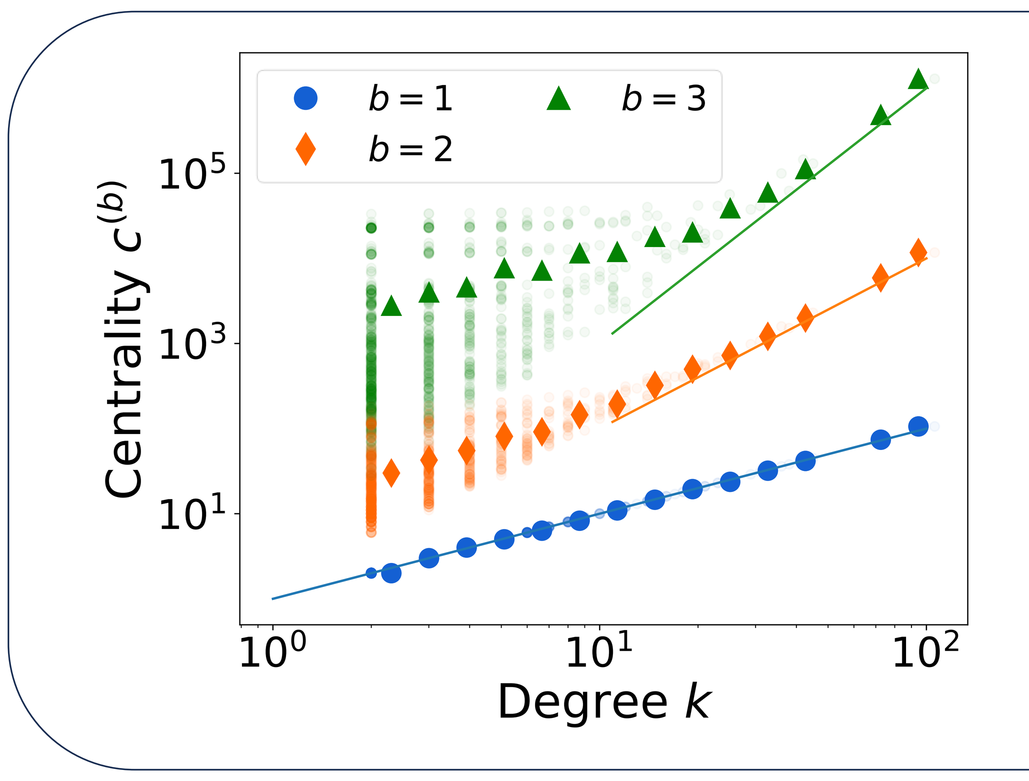

Figure 5: The b-centrality $c_i^{(b)}$ of node i scales like the bth power of the degree k_i of node i, $c_i^{(b)} \propto k^b$ (lines) for large degree nodes $k \gg 1$ but leads to a different ranking for low degree nodes $k \sim 1$. Results for a Barabási-Albert network of $n = 10^3$ nodes and $m = 2 \cdot 10^3$ edges. Filled symbols indicate binning, transparent ones raw data.

Discussion & Open Questions

Blob walks are reminiscent of certain motif counting algorithms [2,3,4] and could help to improve them, but they are interesting dynamical systems in their own right: Ongoing and future research includes classical random walk properties like mean-first passage time and mean squared displacement, but also novel properties like the distribution of blob diameters or visited motifs.

References

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